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VARIATIONAL INTEGRATOR FOR FRACTIONAL EULER-LAGRANGE EQUATIONS.

by

Loïc Bourdin, Jacky Cresson, Isabelle Greff & Pierre Inizan

Abstract. — We extend the notion of variational integrator for classical Euler-Lagrange equations to the fractional ones. As in the classical case, we prove that the variational integrator allows to preserve Noether-type results at the discrete level.

Keywords: Euler-Lagrange equations; fractional calculus; variational integrator; Noether's theorem.

AMS Classification: 70H03; 37K05; 26A33.

1. Introduction

Fractional calculus is the emerging mathematical field dealing with the generalization of the derivative to any real order. During the last two decades, it has been successfully applied to problems in economics [9], computational biology [20] and several fields in Physics [5, 6, 16, 26]. We refer to [17, 23, 25] for a general theory and to [19] for more details concerning the recent history of fractional calculus. Particularly, a subtopic of the fractional calculus has recently gained importance: it concerns the variational principles on functionals involving fractional derivatives. This leads to the statement of fractional Euler-Lagrange equations, see [1, 7, 24].

Fractional Euler-Lagrange equations are difficult to solve explicitly and consequently, it is of interest to develop efficient numerical schemes for such dynamical systems. There exists a suitable method for classical Euler-Lagrange equations called *variational integrator* and well-developed in [15, 21]. The basic idea is to preserve at the discrete level the intrinsic variational structure of the differential equation. In this paper, we extend the notion of variational integrator to the fractional case.

In the classical case, a variational integrator presents many interesting numerical properties as the conservation at the discrete level of constants of motion given by Noether's theorem, see [15, 21] for more details. In this paper, we prove that the variational integrator extended to the fractional case allows to preserve a fractional Noether-type result proved in [10, 14].

The paper is organized as follows. Section 2 is devoted to a reminder concerning fractional calculus and fractional Euler-Lagrange equations. In Section 3, we extend the notion of variational integrator to the fractional case and provide numerical simulations for the fractional Dirichlet example. In Section 4, we remind the fractional Noether-type result proved in [10, 14] and we prove that it is preserved at the discrete level by the variational integrator constructed in the previous section. We conclude this section with the fractional harmonic oscillator example.

2. Reminder about fractional Euler-Lagrange equations

In this paper, we consider fractional differential systems in \mathbb{R}^d where $d \in \mathbb{N}^*$ is the dimension. The trajectories of these systems are curves $q \in \mathcal{C}^0([a, b], \mathbb{R}^d)$ with $a < b$ two reals. Let $0 < \alpha \leq 1$ denote a fractional order and let us define $\alpha_0 := 1$ and for any $r \in \mathbb{N}^*$:

$$\alpha_r := \frac{(-\alpha)(1-\alpha)\dots(r-1-\alpha)}{r!}. \quad (1)$$

2.1. Fractional derivatives of Grünwald-Letnikov. — Fractional calculus deals with the generalization of the usual notion of derivative to any real order. Since 1695, numerous notions of fractional derivatives emerge over the year, see [17, 23, 25]. We will use the fractional operators of Grünwald-Letnikov (1867) whose definitions are reminded below.

Let q be a function defined on $[a, b]$ with values in \mathbb{R}^d . The left (resp. right) fractional derivative in the sense of Grünwald-Letnikov with inferior limit a (resp. superior limit b) of order α of q is given by:

$$\forall t \in]a, b], D_-^\alpha q(t) := \lim_{\substack{h \rightarrow 0^+ \\ nh = t-a}} \frac{1}{h^\alpha} \sum_{r=0}^n \alpha_r q(t - rh), \quad (2)$$

respectively:

$$\forall t \in [a, b[, D_+^\alpha q(t) := \lim_{\substack{h \rightarrow 0^+ \\ nh = b-t}} \frac{1}{h^\alpha} \sum_{r=0}^n \alpha_r q(t + rh), \quad (3)$$

provided the right side terms are defined.

In the classical case $\alpha = 1$, we note that $\alpha_r = 0$ for any $r \geq 2$. Consequently, we obtain that D_-^1 (resp. $-D_+^1$) coincides with the usual notion of left (resp. right) derivative of a function.

We remind the following result used further in the paper:

Lemma 1 (Fractional integration by parts). — *For any $q, w \in \mathcal{C}^1([a, b], \mathbb{R}^d)$ satisfying $w(a) = w(b) = 0$, the following equality holds:*

$$\int_a^b q \cdot D_-^\alpha w \, dt = \int_a^b D_+^\alpha q \cdot w \, dt. \quad (\text{FIBP})$$

Let us recall that the fractional derivatives of Grünwald-Letnikov coincide with the Riemann-Liouville's ones (1832) for smooth enough functions, see [23]. Consequently, we refer to [25, p.46] for a detailed proof of Lemma 1.

2.2. Fractional Euler-Lagrange equations. — Since 1997 (see [24]), a subtopic of fractional calculus gains importance: it concerns the calculus of variations on functionals involving fractional derivatives. It leads to the statement of fractional Euler-Lagrange equations that was developed in [1].

Let L be a *Lagrangian*, i.e. a \mathcal{C}^2 map of the form:

$$\begin{aligned} L : \mathbb{R}^d \times \mathbb{R}^d \times [a, b] &\longrightarrow \mathbb{R} \\ (x, v, t) &\longmapsto L(x, v, t) \end{aligned} \quad (4)$$

and let us consider a *fractional Lagrangian functional* given by:

$$\begin{aligned} \mathcal{L}^\alpha : \mathcal{C}^2([a, b], \mathbb{R}^d) &\longrightarrow \mathbb{R} \\ q &\longmapsto \int_a^b L(q, D_-^\alpha q, t) dt. \end{aligned} \quad (5)$$

Let $\mathcal{C}_0^2([a, b], \mathbb{R}^d) := \{w \in \mathcal{C}^2([a, b], \mathbb{R}^d), w(a) = w(b) = 0\}$ denote the set of *variations* of \mathcal{L}^α . The *critical points* of \mathcal{L}^α are the elements $q \in \mathcal{C}^2([a, b], \mathbb{R}^d)$ satisfying $D\mathcal{L}^\alpha(q)(w) = 0$ for any variations w where:

$$D\mathcal{L}^\alpha(q)(w) := \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{L}^\alpha(q + \varepsilon w) - \mathcal{L}^\alpha(q)}{\varepsilon}. \quad (6)$$

Finally, combining Lemma 1 and an usual calculus of variations, critical points of \mathcal{L}^α are characterized by solutions of the following *fractional Euler-Lagrange equation*:

$$\frac{\partial L}{\partial x}(q, D_-^\alpha q, t) + D_+^\alpha \left(\frac{\partial L}{\partial v}(q, D_-^\alpha q, t) \right) = 0. \quad (\text{EL}^\alpha)$$

We refer to [1] for a detailed proof.

3. Discrete fractional Euler-Lagrange equations

In this section, we are interested in the construction of a variational integrator for fractional Euler-Lagrange equations (EL^α). Then, let us introduce some notations available in the whole paper. Let $N \geq 2$, let $h := (b - a)/N$ be the step size of the discretization and let $\mathbf{T} := (t_k)_{k=0, \dots, N} := (a + kh)_{k=0, \dots, N}$ be the usual partition of the interval $[a, b]$.

3.1. Discrete fractional derivatives of Grünwald-Letnikov. — In order to give a discrete version of a fractional differential equation, we need discrete fractional operators approximating the continuous ones. As it is done in [13], we define the following left (resp. right) discrete fractional derivative of Grünwald-Letnikov:

$$\begin{aligned} \Delta_-^\alpha : (\mathbb{R}^d)^{N+1} &\longrightarrow (\mathbb{R}^d)^N \\ \mathbf{Q} = (Q_k)_{k=0, \dots, N} &\longmapsto \left(\frac{1}{h^\alpha} \sum_{r=0}^k \alpha_r Q_{k-r} \right)_{k=1, \dots, N}, \end{aligned} \quad (7)$$

respectively:

$$\begin{aligned} \Delta_+^\alpha : (\mathbb{R}^d)^{N+1} &\longrightarrow (\mathbb{R}^d)^N \\ \mathbf{Q} = (Q_k)_{k=0, \dots, N} &\longmapsto \left(\frac{1}{h^\alpha} \sum_{r=0}^{N-k} \alpha_r Q_{k+r} \right)_{k=0, \dots, N-1}. \end{aligned} \quad (8)$$

Δ_-^α (resp. Δ_+^α) is an approximation of D_-^α (resp. D_+^α). We refer to [12, 15] for details about these approximations of fractional operators.

In the classical case $\alpha = 1$, we note that Δ_-^1 and $-\Delta_+^1$ coincide respectively with the backward and forward Euler's approximations of the derivative d/dt . Indeed, we have:

$$\begin{aligned} \Delta_-^1 : \quad (\mathbb{R}^d)^{N+1} &\longrightarrow (\mathbb{R}^d)^N \\ \mathbf{Q} = (Q_k)_{k=0,\dots,N} &\longmapsto \left(\frac{Q_k - Q_{k-1}}{h} \right)_{k=1,\dots,N}, \end{aligned} \quad (9)$$

and:

$$\begin{aligned} \Delta_+^1 : \quad (\mathbb{R}^d)^{N+1} &\longrightarrow (\mathbb{R}^d)^N \\ \mathbf{Q} = (Q_k)_{k=0,\dots,N} &\longmapsto \left(\frac{Q_k - Q_{k+1}}{h} \right)_{k=0,\dots,N-1}. \end{aligned} \quad (10)$$

3.2. Remark about direct discretization and variational integrator. — An usual and algebraic way to obtain a discrete version of a fractional differential equation is to replace the curves $q \in \mathcal{C}^0([a, b], \mathbb{R}^d)$ by discrete elements $\mathbf{Q} \in (\mathbb{R}^d)^{N+1}$ and to replace fractional derivatives D_\pm^α by discrete fractional derivatives Δ_\pm^α . This method is widely used and one can find examples in [4, 18, 22] for fractional differential equations and fractional partial differential equations. Other examples can be found in [2, 3, 11] for fractional optimal control problems.

Such a discretization leads to the following numerical scheme for a fractional Euler-Lagrange equation (EL^α) :

$$\forall k = 1, \dots, N-1, \quad \frac{\partial L}{\partial x}(Q_k, (\Delta_-^\alpha \mathbf{Q})_k, t_k) + \Delta_+^\alpha \left(\frac{\partial L}{\partial v}(\mathbf{Q}, \Delta_-^\alpha \mathbf{Q}, \mathbf{T}) \right)_k = 0. \quad (11)$$

However, a fractional Euler-Lagrange equation (EL^α) admits a variational structure in the sense that it derives from a variational principle on a functional, see Section 2.2. This structure is intrinsic and induces strong constraints on the qualitative behaviour of the solutions. It seems then important to preserve this structure at the discrete level. Nevertheless, the numerical scheme (11) is obtained by an *algebraic* procedure only based on the differential writing of (EL^α) . Consequently, there are no guarantees that the intrinsic variational structure of (EL^α) is preserved at the discrete level.

In Section 3.3, we are going to construct a *variational integrator* for fractional Euler-Lagrange equations (EL^α) . This procedure has a variational approach and allows to preserve the variational structure of (EL^α) at the discrete level.

Let us give more details concerning the construction of a variational integrator. Let us consider a differential system admitting a variational structure (*i.e.* deriving from a variational principle on a functional and then, characterizing its critical points). A variational integrator is a numerical scheme constructed as follows:

- the first step consists in defining a discrete version of the functional;
- the second step consists in forming a discrete variational principle on the discrete functional characterizing its discrete critical points.

Hence, a numerical scheme is obtained and it is called variational integrator. The variational structure of the differential system is then preserved at the discrete level in the sense that the discrete solutions correspond to the discrete critical points of the discrete version of the initial functional.

We recall that a variational integrator is well-studied for classical Euler-Lagrange equations ($\alpha = 1$) in [15, 21]. In this case, the conservation of the variational structure at the discrete level allows to preserve properties and results relative to this structure. As an example, we can cite Noether's theorem. In the following section, our aim is to extend this variational integrator to the fractional case ($0 < \alpha \leq 1$).

3.3. Construction of a variational integrator. — In this section, we construct a variational integrator for fractional Euler-Lagrange equations following the two steps described in the previous section.

First step. — Let us define a discrete version of the fractional Lagrangian functional \mathcal{L}^α . Considering the usual Gaussian quadrature formula, we define the following *discrete fractional Lagrangian functional*:

$$\begin{aligned} \mathcal{L}_h^\alpha : (\mathbb{R}^d)^{N+1} &\longrightarrow \mathbb{R} \\ \mathbf{Q} &\longmapsto h \sum_{k=1}^N L(Q_k, (\Delta_-^\alpha \mathbf{Q})_k, t_k). \end{aligned} \quad (12)$$

Hence, the first step of construction of a variational integrator for (EL^α) is completed.

Second step. — Let $(\mathbb{R}^d)_0^{N+1} := \{\mathbf{W} \in (\mathbb{R}^d)^{N+1}, W_0 = W_N = 0\}$ denote the set of *discrete variations* of \mathcal{L}_h^α . In the sequel, we focus on *discrete critical points* of \mathcal{L}_h^α , i.e. elements $\mathbf{Q} \in (\mathbb{R}^d)^{N+1}$ satisfying $D\mathcal{L}_h^\alpha(\mathbf{Q})(\mathbf{W}) = 0$ for any discrete variations \mathbf{W} where:

$$D\mathcal{L}_h^\alpha(\mathbf{Q})(\mathbf{W}) := \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{L}_h^\alpha(\mathbf{Q} + \varepsilon \mathbf{W}) - \mathcal{L}_h^\alpha(\mathbf{Q})}{\varepsilon}. \quad (13)$$

As in the continuous case, in order to obtain a characterization of the discrete critical points of \mathcal{L}_h^α , we first need a preliminary result. We prove the following discrete version of Lemma 1:

Lemma 2 (Discrete fractional integration by parts). — *For any $\mathbf{Q}, \mathbf{W} \in (\mathbb{R}^d)^{N+1}$ satisfying $W_0 = W_N = 0$, the following equality holds:*

$$h \sum_{k=1}^N Q_k \cdot (\Delta_-^\alpha \mathbf{W})_k = h \sum_{k=0}^{N-1} (\Delta_+^\alpha \mathbf{Q})_k \cdot W_k. \quad (\text{DFIBP})$$

Proof. — Since $W_0 = W_N = 0$, the following equalities hold:

$$\begin{aligned}
h^\alpha \sum_{k=1}^N Q_k \cdot (\Delta_-^\alpha \mathbf{W})_k &= \sum_{k=1}^N \sum_{r=0}^k \alpha_r Q_k \cdot W_{k-r} = \sum_{k=0}^N \sum_{r=0}^k \alpha_r Q_k \cdot W_{k-r} \\
&= \sum_{r=0}^N \sum_{k=r}^N \alpha_r Q_k \cdot W_{k-r} = \sum_{r=0}^N \sum_{k=0}^{N-r} \alpha_r Q_{k+r} \cdot W_k \\
&= \sum_{k=0}^N \sum_{r=0}^k \alpha_r Q_{k+r} \cdot W_k = \sum_{k=0}^N \sum_{r=0}^{N-1} \alpha_r Q_{k+r} \cdot W_k \\
&= h^\alpha \sum_{k=0}^N (\Delta_+^\alpha \mathbf{Q})_k \cdot W_k.
\end{aligned} \tag{14}$$

Multiplying by $h^{1-\alpha}$, the proof is complete. \square

Finally, combining Lemma 2 and a discrete calculus of variations, we obtain the following result:

Theorem 1. — *Let $\mathbf{Q} \in (\mathbb{R}^d)^{N+1}$. Then, \mathbf{Q} is a discrete critical point of \mathcal{L}_h^α if and only if \mathbf{Q} is solution of the following discrete fractional Euler-Lagrange equation:*

$$\forall k = 1, \dots, N-1, \quad \frac{\partial L}{\partial x}(Q_k, (\Delta_-^\alpha \mathbf{Q})_k, t_k) + \Delta_+^\alpha \left(\frac{\partial L}{\partial v}(\mathbf{Q}, \Delta_-^\alpha \mathbf{Q}, \mathbf{T}) \right)_k = 0. \tag{EL}_h^\alpha$$

Proof. — Let $\mathbf{Q} \in (\mathbb{R}^d)^{N+1}$ and $\mathbf{W} \in (\mathbb{R}^d)_0^{N+1}$. Let us define the following function:

$$\begin{aligned}
\varphi : \mathbb{R} &\longrightarrow \mathbb{R} \\
\varepsilon &\longmapsto \mathcal{L}_h^\alpha(\mathbf{Q} + \varepsilon \mathbf{W}) = h \sum_{k=1}^N L(Q_k + \varepsilon W_k, (\Delta_-^\alpha \mathbf{Q})_k + \varepsilon (\Delta_-^\alpha \mathbf{W})_k, t_k).
\end{aligned} \tag{15}$$

Then, since $D\mathcal{L}_h^\alpha(\mathbf{Q})(\mathbf{W}) = \dot{\varphi}(0)$, we have:

$$D\mathcal{L}_h^\alpha(\mathbf{Q})(\mathbf{W}) = h \sum_{k=1}^N \left[\frac{\partial L}{\partial x}(Q_k, (\Delta_-^\alpha \mathbf{Q})_k, t_k) \cdot W_k + \frac{\partial L}{\partial v}(Q_k, (\Delta_-^\alpha \mathbf{Q})_k, t_k) \cdot (\Delta_-^\alpha \mathbf{W})_k \right]. \tag{16}$$

Since $\mathbf{W} \in (\mathbb{R}^d)_0^{N+1}$ and using the discrete fractional integration by parts given in Lemma 2, we obtain:

$$D\mathcal{L}_h^\alpha(\mathbf{Q})(\mathbf{W}) = h \sum_{k=1}^{N-1} \left[\frac{\partial L}{\partial x}(Q_k, (\Delta_-^\alpha \mathbf{Q})_k, t_k) + \Delta_+^\alpha \left(\frac{\partial L}{\partial v}(\mathbf{Q}, \Delta_-^\alpha \mathbf{Q}, \mathbf{T}) \right)_k \right] \cdot W_k, \tag{17}$$

which completes the proof. \square

Theorem 1 completes the second step. Precisely, the discrete fractional Euler-Lagrange equation (EL_h^α) is the variational integrator constructed for the fractional Euler-Lagrange equation (EL^α) . It is a numerical scheme preserving the variational structure of (EL^α) in the sense that the discrete solutions of (EL_h^α) coincide with the discrete critical points of the discrete version \mathcal{L}_h^α of \mathcal{L}^α .

Remark 1. — *We note that the discrete Euler-Lagrange equation (EL_h^α) coincides with the numerical scheme (11) obtained with a direct discretization. It is important to note that such a phenomena is not obvious. Indeed, in the classical case $\alpha = 1$, we have $D_-^\alpha = -D_+^\alpha = d/dt$*

and then, it is not obvious to know how to replace d/dt at the discrete level. Indeed, one can choose Δ_-^1 or $-\Delta_+^1$ or a mixing of the two of them. In the fractional case, the non locality of the fractional operators does not permit such a choice.

In the classical case $\alpha = 1$, we note that (EL_h^α) coincides with the classical discrete Euler-Lagrange equation provided in literature about variational integrators, see [15, 21].

3.4. The fractional Dirichlet example. — In general, the fractional Euler-Lagrange equations are very difficult to solve explicitly. There are only few examples where exact solutions are provided and furthermore not in an explicit way. Due to the additional complexity of fractional operators, the purpose of this section is only to obtain experimental results and not to provide numerical analyses.

In this section, we consider the fractional Dirichlet example studied in [1] and where a *quasi-explicit* solution is known. Precisely, we consider the interval $[a, b] = [0, 1]$, $d = 1$ and the following Lagrangian:

$$\begin{aligned} L : \mathbb{R} \times \mathbb{R} \times [0, 1] &\longrightarrow \mathbb{R} \\ (x, v, t) &\longmapsto \frac{1}{2}v^2. \end{aligned} \quad (18)$$

The associated fractional Euler-Lagrange equation (EL^α) is given by $D_+^\alpha \circ D_-^\alpha q = 0$. For any $1/2 < \alpha \leq 1$, under boundary assumptions $q(0) = 0$ and $q(1) = 1$, this equation admits a unique solution given by:

$$q(t) = (2\alpha - 1) \int_0^t \frac{1}{[(1-x)(t-x)]^{1-\alpha}} dx. \quad (19)$$

We note that for $\alpha = 1$, the unique solution is $q(t) = t$. Nevertheless, in the strict fractional case $1/2 < \alpha < 1$, q is not explicitly given since it is written with an improper integral. As a consequence, the following numerical results consider an approximation of the exact solution q using high-order global adaptive quadrature.

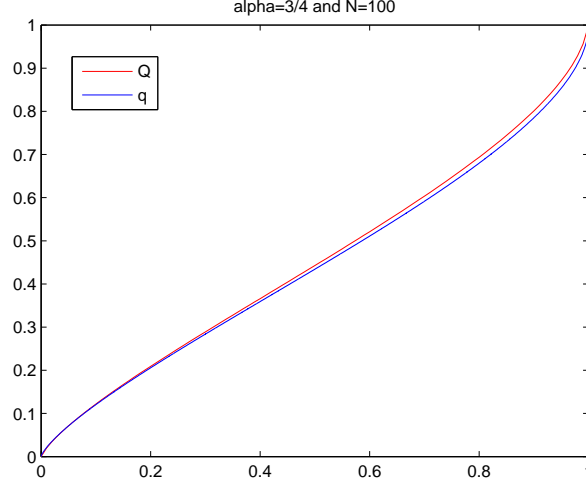
Let us solve the discrete fractional Euler-Lagrange equation (EL_h^α) given by:

$$\forall k = 1, \dots, N-1, (\Delta_+^\alpha \circ \Delta_-^\alpha \mathbf{Q})_k = 0, \quad (20)$$

with boundary values $Q_0 = 0$ and $Q_N = 1$. For any $1/2 < \alpha \leq 1$ and any $N \geq 2$, the ℓ^∞ -error and the ℓ^2 -error between the exact solution q and the discrete one \mathbf{Q} are defined by:

$$E_{\alpha, N}^\infty = \max_{k=0, \dots, N} |Q_k - q(t_k)| \text{ and } E_{\alpha, N}^2 = \sqrt{h \sum_{k=0}^N |Q_k - q(t_k)|^2}, \quad (21)$$

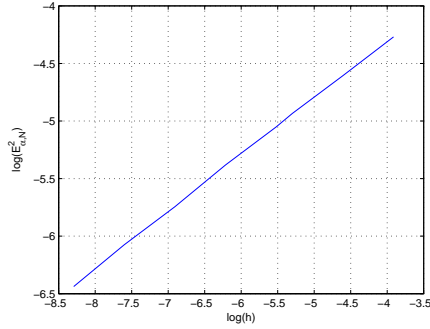
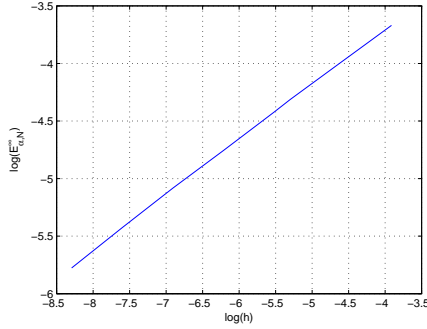
where $h = 1/N$. We compare q and Q for $\alpha = 3/4$ and $N = 100$. The solutions are displayed on the following picture:



The errors $E_{\alpha,N}^\infty$ and $E_{\alpha,N}^2$ for $\alpha = 3/4$ and for varying N are given in the following table:

N	50	100	200	250	500	1000	2000	4000
$E_{3/4,N}^\infty$	0.0255	0.0185	0.0134	0.0120	0.0086	0.0062	0.0044	0.0031
$E_{3/4,N}^2$	0.0140	0.0100	0.0072	0.0064	0.0046	0.0032	0.0023	0.0016

Finally, the graphic representations of $\log(E_{3/4,N}^\infty)$ and $\log(E_{3/4,N}^2)$ with respect to $\log(h)$ are respectively given on the following pictures:



We clearly obtain two linear functions with the common slope $\lambda_{3/4} \simeq 0.51$. In this case, we conclude that we obtain an experimental convergence of order $\lambda_{3/4} \simeq 0.51$.

Now, let us study the evolution of the errors $E_{\alpha,N}^\infty$ and $E_{\alpha,N}^2$ and the slope λ_α with respect to α . Let us precise that the numerical study of the case $\alpha = 1$ is not interesting because, in this case, the discrete Euler-Lagrange equation (EL_h^1) gives directly the exact solution $q(t) = t$. In the strict fractional case $1/2 < \alpha < 1$, the errors $E_{\alpha,N}^\infty$ and $E_{\alpha,N}^2$ are given in the following table:

$\alpha \backslash N$	50	100	200	250	500	1000	2000	4000
0.55	0.3811	0.3679	0.3527	0.3475	0.3310	0.3141	0.2974	0.2810
	0.1746	0.1560	0.1393	0.1344	0.1203	0.1079	0.0972	0.0878
0.6	0.1988	0.1788	0.1596	0.1537	0.1363	0.1204	0.1061	0.0933
	0.0969	0.0825	0.0701	0.0665	0.0566	0.0483	0.0413	0.0354
0.65	0.1023	0.0856	0.0711	0.0669	0.0552	0.0453	0.0372	0.0304
	0.0525	0.0423	0.0340	0.0317	0.0255	0.0205	0.0165	0.0133
0.7	0.0517	0.0403	0.0311	0.0286	0.0219	0.0168	0.0129	0.0098
	0.0276	0.0210	0.0159	0.0146	0.0110	0.0083	0.0063	0.0048
0.75	0.0255	0.0185	0.0134	0.0120	0.0086	0.0062	0.0044	0.0031
	0.0140	0.0100	0.0072	0.0064	0.0046	0.0032	0.0023	0.0016
0.8	0.0122	0.0083	0.0056	0.0049	0.0033	0.0022	0.0015	0.00097
	0.0067	0.0045	0.0030	0.0027	0.0018	0.0012	0.0008	0.00052
0.85	0.0055	0.0035	0.0022	0.0019	0.0012	0.00074	0.00046	0.00028
	0.0030	0.0019	0.0012	0.0010	0.0006	0.0004	0.00025	0.00015
0.9	0.0022	0.0013	0.00079	0.00066	0.00039	0.00022	0.00013	0.000075
	0.0012	0.0007	0.00041	0.00035	0.0002	0.00012	0.00007	0.00004
0.95	0.00068	0.00038	0.00021	0.00017	0.000095	0.000052	0.000028	0.000015
	0.00037	0.0002	0.00011	0.00009	0.00005	0.000026	0.000014	0.000007

We clearly see that the more α is close to 1, the more the approximation is sensibly better. Now, we study the evolution of slope λ_α with respect to α :

α	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
λ_α	0.11	0.2	0.3	0.39	0.51	0.62	0.72	0.8	0.91

For this example, we conclude that we obtain an experimental convergence of order $\lambda_\alpha = 2\alpha - 1$.

4. Conservation of a fractional Noether-type result

It is well-known that the conservation at the discrete level of the variational structure allows to preserve some properties relative to this structure. For example, in the classical case $\alpha = 1$, we know that the classical Noether's theorem is preserved, see [15, 21]. In this section, in the fractional case $0 < \alpha \leq 1$, we prove the conservation at the discrete level of the fractional Noether-type result proved simultaneously by Cresson [10] and Torres *et al.* [14].

4.1. Reminder about a fractional Noether-type theorem. — We first review the definition of a one parameter group of diffeomorphisms:

Definition 1. — For any real s , let $\phi(s, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a diffeomorphism. Then, $\Phi = \{\phi(s, \cdot)\}_{s \in \mathbb{R}}$ is a one parameter group of diffeomorphisms if it satisfies

1. $\phi(0, \cdot) = Id_{\mathbb{R}^d}$,
2. $\forall s, s' \in \mathbb{R}, \phi(s, \cdot) \circ \phi(s', \cdot) = \phi(s + s', \cdot)$,
3. ϕ is of class \mathcal{C}^2 .

Classical examples of one parameter groups of diffeomorphisms are given by translations and rotations.

The action of a one parameter group of diffeomorphisms on a Lagrangian allows to define the notion of a symmetry for a fractional Euler-Lagrange equation:

Definition 2. — Let $\Phi = \{\phi(s, \cdot)\}_{s \in \mathbb{R}}$ be a one parameter group of diffeomorphisms and let L be a Lagrangian. L is said to be D_-^α -invariant under the action of Φ if it satisfies:

$$\forall q \text{ solution of } (\text{EL}^\alpha), \forall s \in \mathbb{R}, L(\phi(s, q), D_-^\alpha(\phi(s, q)), t) = L(q, D_-^\alpha q, t). \quad (22)$$

For example, any quadratic Lagrangian is D_-^α -invariant under the action of rotations. We refer to Section 4.3 for the fractional harmonic oscillator example. We also note that the fractional Dirichlet example studied in Section 3.4 is associated to a quadratic Lagrangian and then, it admits a symmetry.

With this definition of symmetry, Cresson [10] and Torres *et al.* [14] proved the following result:

Theorem 2 (Fractional Noether-type theorem). — Let L be a Lagrangian D_-^α -invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s, \cdot)\}_{s \in \mathbb{R}}$. Then, the following equality holds for any solution q of (EL^α) :

$$D_-^\alpha \left(\frac{\partial \phi}{\partial s}(0, q) \right) \cdot \frac{\partial L}{\partial v}(q, D_-^\alpha q, t) - \frac{\partial \phi}{\partial s}(0, q) \cdot D_+^\alpha \left(\frac{\partial L}{\partial v}(q, D_-^\alpha q, t) \right) = 0. \quad (23)$$

In the classical case $\alpha = 1$, the classical Leibniz formula allows to rewrite (23) as the derivative of a product. Precisely, Theorem 2 leads to the classical Noether's theorem given by:

Theorem 3 (Classical Noether's theorem). — Let L be a Lagrangian d/dt -invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s, \cdot)\}_{s \in \mathbb{R}}$. Then, the following equality holds for any solution q of (EL^1) :

$$\frac{d}{dt} \left(\frac{\partial \phi}{\partial s}(0, q) \cdot \frac{\partial L}{\partial v}(q, \dot{q}, t) \right) = 0, \quad (24)$$

where \dot{q} is the classical derivative of q , i.e. dq/dt .

Theorem 3 provides an explicit constant of motion for any classical Euler-Lagrange equations (EL^1) admitting a symmetry. In the fractional case, such a simple Leibniz formula allowing to rewrite (23) as a total derivative with respect to t is not known yet. Nevertheless, from Theorem 2, we prove in [8] the existence of an explicit constant of motion written with an infinite sum (this result is proved *via* an iterative application of Leibniz formulas). For sake of brevity, we do not develop this result in this paper.

4.2. A discrete fractional Noether-type theorem. — In this section, we prove that Theorem 2 is preserved at the discrete level with the discrete fractional Euler-Lagrange equations (EL_h^α) .

As in the continuous case, a discrete symmetry of a discrete fractional Euler-Lagrange equation is based on the action of a one parameter group of transformations on the associated Lagrangian:

Definition 3. — Let $\Phi = \{\phi(s, \cdot)\}_{s \in \mathbb{R}}$ be a one parameter group of diffeomorphisms and let L be a Lagrangian. L is said to be Δ_-^α -invariant under the action of Φ if it satisfies:

$$\forall \mathbf{Q} \text{ solution of } (\text{EL}_h^\alpha), \forall s \in \mathbb{R}, L(\phi(s, \mathbf{Q}), \Delta_-^\alpha(\phi(s, \mathbf{Q})), T) = L(\mathbf{Q}, \Delta_-^\alpha \mathbf{Q}, T). \quad (25)$$

As in the continuous case, any quadratic Lagrangian is Δ_-^α -invariant under the action of rotations. We refer to Section 4.3 for the fractional harmonic oscillator example. The fractional Dirichlet example studied in Section 3.4 is also associated to a quadratic Lagrangian and then, it admits a discrete symmetry.

From this notion of discrete symmetry, we prove the discrete analogous version of Theorem 2:

Theorem 4 (Discrete fractional Noether-type theorem). — Let L be a Lagrangian Δ_-^α -invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s, \cdot)\}_{s \in \mathbb{R}}$. Then, the following equality holds for any solution \mathbf{Q} of (EL_h^α) :

$$\Delta_-^\alpha \left(\frac{\partial \phi}{\partial s}(0, \mathbf{Q}) \right) \cdot \frac{\partial L}{\partial v}(\mathbf{Q}, \Delta_-^\alpha \mathbf{Q}, T) - \frac{\partial \phi}{\partial s}(0, \mathbf{Q}) \cdot \Delta_+^\alpha \left(\frac{\partial L}{\partial v}(\mathbf{Q}, \Delta_-^\alpha \mathbf{Q}, T) \right) = 0. \quad (26)$$

Proof. — This proof is a direct adaptation to the discrete case of the proof of Theorem 2. Let $\mathbf{Q} \in (\mathbb{R}^d)^{N+1}$ be a solution of (EL_h^α) . Let us differentiate equation (25) with respect to s and invert the operators Δ_-^α and $\partial/\partial s$. We finally obtain for any $s \in \mathbb{R}$ and any $k \in \{1, \dots, N-1\}$:

$$\begin{aligned} \Delta_-^\alpha \left(\frac{\partial \phi}{\partial s}(s, \mathbf{Q}) \right)_k \cdot \frac{\partial L}{\partial v}(\phi(s, \mathbf{Q}_k), \Delta_-^\alpha(\phi(s, \mathbf{Q}_k)), t_k) \\ + \frac{\partial L}{\partial x}(\phi(s, \mathbf{Q}_k), \Delta_-^\alpha(\phi(s, \mathbf{Q}_k)), t_k) \cdot \frac{\partial \phi}{\partial s}(s, \mathbf{Q}_k) = 0. \end{aligned} \quad (27)$$

Since $\phi(0, \cdot) = Id_{\mathbb{R}^d}$ and \mathbf{Q} is solution of (EL_h^α) , taking $s = 0$ in (27) leads to (26). \square

From this last result, we conclude that the discrete fractional Euler-Lagrange equation (EL_h^α) allows to preserve the fractional Noether's Theorem 2 at the discrete level.

We recall the following discrete Leibniz formula:

$$\forall \mathbf{F}, \mathbf{G} \in (\mathbb{R}^d)^{N+1}, \forall k = 1, \dots, N-1, \Delta_-^1(\mathbf{F} \cdot \sigma(\mathbf{G}))_k = (\Delta_-^1 \mathbf{F})_k \cdot G_k - F_k \cdot (\Delta_+^1 \mathbf{G})_k, \quad (28)$$

where $\sigma(\mathbf{G})_k = G_{k+1}$ for any $k = 0, \dots, N-1$. From this discrete Leibniz formula, we note that Theorem 4 takes a particular simple expression in the classical case $\alpha = 1$. It corresponds to the discrete version of Theorem 3:

Theorem 5 (Discrete classical Noether's theorem). — Let L be a Lagrangian Δ_-^1 -invariant under the action of a one parameter group of diffeomorphisms $\Phi = \{\phi(s, \cdot)\}_{s \in \mathbb{R}}$. Then, the following equality holds for any solution \mathbf{Q} of (EL_h^1) :

$$\Delta_-^1 \left[\frac{\partial \phi}{\partial s}(0, \mathbf{Q}) \cdot \sigma \left(\frac{\partial L}{\partial v}(\mathbf{Q}, \Delta_-^1 \mathbf{Q}, T) \right) \right] = 0. \quad (29)$$

This result is a reformulation of the discrete Noether's theorem proved in [15, 21]. We recall the following implication:

$$\forall \mathbf{F} \in \mathbb{R}^{N+1}, \Delta_-^1 \mathbf{F} = 0 \implies \exists c \in \mathbb{R}, \forall k = 0, \dots, N, F_k = c. \quad (30)$$

Consequently, as in the continuous case, Theorem 5 provides an explicit discrete constant of motion for any discrete classical Euler-Lagrange equations (EL_h^1) admitting a symmetry. In the fractional case, such a simple Leibniz formula allowing to rewrite (26) as a total discrete derivative is not known yet. Nevertheless, from Theorem 4, we prove in [8] the existence of a discrete constant of motion computable in a finite number of steps. For sake of brevity, we do not develop this result in this paper.

4.3. The fractional harmonic oscillator example. — We conclude this section with the fractional harmonic oscillator example where Noether-type Theorems 2 and 4 are available. Precisely, we consider $[a, b] = [0, 1]$, $d = 2$ and the following quadratic Lagrangian:

$$\begin{aligned} L : \mathbb{R}^2 \times \mathbb{R}^2 \times [0, 1] &\longrightarrow \mathbb{R} \\ (x, v, t) &\longmapsto \frac{1}{2}(\|x\|^2 + \|v\|^2). \end{aligned} \quad (31)$$

Since L is quadratic, for any $0 < \alpha \leq 1$, L is D_-^α -invariant and Δ_-^α -invariant under the action of rotations given by:

$$\begin{aligned} \phi : \mathbb{R} \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (s, x_1, x_2) &\longmapsto \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned} \quad (32)$$

In the classical continuous case $\alpha = 1$, the Euler-Lagrange equation (EL^1) is given by $\ddot{q} = q$ and consequently, the exact solutions are given by $q(t) = (q^1(t), q^2(t)) = (c_1 e^t + c_2 e^{-t}, c_3 e^t + c_4 e^{-t})$ for any $t \in [0, 1]$ where $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Applying Theorem 3, we obtain an explicit constant of motion. Indeed, for any $q = (q^1, q^2)$ solution of (EL^1) , there exists $c \in \mathbb{R}$ such that:

$$\frac{\partial \phi}{\partial s}(0, q) \cdot \frac{\partial L}{\partial v}(q, \dot{q}, t) = (-q^2, q^1) \cdot (\dot{q}^1, \dot{q}^2) = c. \quad (33)$$

Similarly, the discrete Euler-Lagrange equation (EL_h^1) is given by $-\Delta_+^1 \circ \Delta_-^1 Q = Q$ and Theorem 5 leads to an explicit discrete constant of motion. Indeed, for any $Q = (Q^1, Q^2)$ discrete solution of (EL_h^1) , there exists $c \in \mathbb{R}$ such that:

$$\frac{\partial \phi}{\partial s}(0, Q) \cdot \sigma \left(\frac{\partial L}{\partial v}(Q, \Delta_-^1 Q, T) \right) = (-Q^2, Q^1) \cdot \sigma(\Delta_-^1 Q^1, \Delta_-^1 Q^2) = c. \quad (34)$$

In the strict fractional continuous case $0 < \alpha < 1$, the fractional Euler-Lagrange equation (EL^α) is given by $q + D_+^\alpha \circ D_-^\alpha q = 0$. Even if explicit solutions are not known yet, the fractional Noether-type Theorem 2 is available and leads to an explicit constant of motion in [8]. In the discrete case, the discrete fractional Euler-Lagrange equation (EL_h^α) is given by $Q + \Delta_+^\alpha \circ \Delta_-^\alpha Q = 0$. The discrete solutions are computable and the discrete fractional Noether-type Theorem 4 (leading to an explicit discrete constant of motion in [8]) is available.

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